MMAT5390: Mathematical Image Processing

Assignment 2

Due: 23:59 Monday, February 24, 2025

Please give reasons in your solutions unless otherwise specified.
In the following, we use $\mathbb{N}$ to denote the set containing all natural numbers starting from 0.

1. Let $H_n(t)$ be the n^{th} Haar function, where $n \in \mathbb{N}$.
   (a) Give the definition of $H_n(t)$, and derive the Haar transform matrix for 4×4 images.
   (b) Let $A = \begin{pmatrix} 2 & 4 & 6 & 8 \\ 4 & 6 & 8 & 0 \\ 6 & 8 & 0 & 4 \\ 8 & 0 & 4 & 2 \end{pmatrix}$. Using the Haar transform matrix derived above to compute the Haar transform A_{Haar} of A.
   (c) By setting values of the smallest (in absolute value) nonzero elements of A_{Haar} to 0, we obtain $\tilde{A}_{\text{Haar}}$. Compute the reconstructed image $\tilde{A}$ of $\tilde{A}_{\text{Haar}}$.

2. (a) Give the definition of 2D DFT of an $M \times N$ image, and write down the Fourier transform matrix U for 4×4 images.
   (b) Let $B = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$. Compute the discrete Fourier transform B_{DFT} of B.
   (c) By setting values of the smallest (in absolute value) nonzero elements of B_{DFT} to 0, we obtain $\tilde{B}_{\text{DFT}}$. Compute the reconstructed image $\tilde{B}$ of $\tilde{B}_{\text{DFT}}$.

3. (a) Use the Fourier transform matrix U obtained in last question to compute the DFT of the following image:
$$g = \begin{pmatrix} 2 & 0 & 2 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
   (b) Suppose there is another image $f \in \mathbb{R}^{4\times 4}$ such that
$$\widehat{f * g} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
   Find f.

4. Let $g = g(k,l)$ where $0 \le k, l \le N-1$ be an $N \times N$ image, and denote its reflection about $(-\frac{1}{2}, -\frac{1}{2})$ by $\tilde{g} = \tilde{g}(k,l)$ where $-N \le k, l \le -1$. That is,
$$\tilde{g}(k,l) = g(-1-k, -1-l) \text{ for } -N \le k, l \le -1.$$
   Prove that the discrete Fourier transform (DFT) of $\tilde{g}$ is given by:
$$DFT(\tilde{g})(m,n) = e^{2\pi j \frac{m+n}{N}} \hat{g}(-m,-n),$$
   where $\hat{g}$ is the DFT of g.

5. (Optional) Let $H_n(t)$ denote the n^{th} Haar function, where $n \in \mathbb{N}$. For any functions $f, g \in L^2(\mathbb{R})$, we denote their inner product by

$$\langle f, g \rangle = \int_{\mathbb{R}} fg,$$

where

$$L^2(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R} \,\middle|\, \int_{\mathbb{R}} f^2 < \infty \right\},$$

(a) (Unit) Prove that $\displaystyle\int_{\mathbb{R}} [H_m(t)]^2 \, dt = 1$ for any $m \in \mathbb{N}$.

(Hence $H_m \in L^2(\mathbb{R})$ and $\|H_m\| = 1$.)

(b) (Orthogonality)

i. Prove that $\langle H_0, H_m \rangle = 0$ for any $m \in \mathbb{N} \setminus \{0\}$.

ii. Let $m_1, m_2 \in \mathbb{N}$ such that $0 \neq m_1 < m_2$. Then $m_1 = 2^{p_1} + n_1$ and $m_2 = 2^{p_2} + n_2$ for some $p_1, p_2 \in \mathbb{N}$, $n_1 \in \mathbb{Z} \cap [0, 2^{p_1} - 1]$ and $n_2 \in \mathbb{Z} \cap [0, 2^{p_2} - 1]$.

A. Suppose $p_1 = p_2$. Prove that $\langle H_{m_1}, H_{m_2} \rangle = 0$.
Hint. In this case $n_1 < n_2$.

B. Suppose $p_1 < p_2$.

- Show that the length of $\left[0, \dfrac{n_1}{2^{p_1}}\right)$ is a multiple of that of $\left[\dfrac{n_2}{2^{p_2}}, \dfrac{n_2 + 1}{2^{p_2}}\right)$.
- Show that the length of $\left[\dfrac{n_1}{2^{p_1}}, \dfrac{n_1 + 0.5}{2^{p_1}}\right)$ is a multiple of that of $\left[\dfrac{n_2}{2^{p_2}}, \dfrac{n_2 + 1}{2^{p_2}}\right)$.
- Show that the length of $\left[\dfrac{n_1 + 0.5}{2^{p_1}}, \dfrac{n_1 + 1}{2^{p_1}}\right)$ is a multiple of that of $\left[\dfrac{n_2}{2^{p_2}}, \dfrac{n_2 + 1}{2^{p_2}}\right)$.
- According to the above steps, considering the possible subset relations between the supports of H_{m_1} and H_{m_2}, prove that $\langle H_{m_1}, H_{m_2} \rangle = 0$.